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Cyclotomic Shuffles[1]

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Abstract—Analogues of 1-shuffle elements for complex reflection groups of type $G(m,1,n)$ are introduced. A geometric interpretation for $G(m,1,n)$ in terms of rotational permutations of polygonal cards is given. We compute the eigenvalues, and their multiplicities, of the 1-shuffle element in the algebra of the group $G(m,1,n)$. Considering shuffling as a random walk on the group $G(m,1,n)$, we estimate the rate of convergence to randomness of the corresponding Markov chain. We report on the spectrum of the 1-shuffle analogue in the cyclotomic Hecke algebra $H(m,1,n)$ for $m = 2$ and small n.

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INTRODUCTION

In 1988, N. Wallach considered an element of the group algebra of the symmetric group S_n which is the sum of cycles $(12\dots i)$ where i ranges from 1 to n. He discovered in [20] that the operator of the left multiplication by this element in the group algebra $\mathbb{Z}S_n$ is diagonalizable with eigenvalues

$$0, 1, 2, \dots, n-2, n. \qquad (1)$$

The sum of cycles $(12\dots i)$ denoted $Ш_{1,n-1}$ appears in different circumstances and is called 1-shuffle element. In particular, it describes all possible ways of removing the top card from the deck of n cards and inserting it back in the deck at a random position.

Investigating the repeated top-to-random shuffling as a random walk on S_n, P. Diaconis et al. [5] (see also R. Phatarfod [18]) found that the multiplicity of the eigenvalue i in (1) is equal to the number of permutations in S_n with i fixed points, explaining the absence of $n-1$ in (1).

The q-deformation of the result of N. Wallach for the q-analogue ${}^{q}Ш_{1,n-1}$ in the Hecke algebra $H_n(q)$ was proposed by G. Lusztig in [11]. He established that the spectrum of the operator $L_{Ш_{1,n-1}}$ of the left multiplication by $Ш_{1,n-1}$ consists of the q-numbers

$$q^{j-1}[j]_q := 1 + q^2 + q^4 + \dots + q^{2j-2}, \\ j = 0, 1, \dots, n-2, n. \qquad (2)$$

Later, A. Isaev and O. Ogievetsky considered shuffle elements $Ш_{p,q}$ in the braid group ring $\mathbb{Z}B_n$. With the help of baxterized elements [9], they constructed additive and multiplicative analogues of $Ш_{p,q}$ in Hecke and Birman–Murakami–Wenzl algebras. The multiplicities of the eigenvalues in (2) have been established therein by taking the trace of $L_{Ш_{1,n-1}} : H_n(q) \to H_n(q)$, using the fact that the q-numbers (2) are linearly independent over $\mathbb{Z}$ as polynomials in q. For generic q, the multiplicities turn out to be the same as for the symmetric group.

In the present paper we propose polygonal analogues of cards that we call m-cards. We introduce elements ${}^{(m)}Ш_{1,n-1}$, which realise the analogues of top to random shuffling on m-cards. The elements ${}^{(m)}Ш_{1,n-1}$ belong to the group algebra of complex reflection groups of type $G(m,1,n)$. We adopt the approach of [5] (for details see [6]) and of [9] to compute the spectrum and the multiplicities of the eigenvalues of $L_{{}^{(m)}Ш_{1,n-1}}$. The obtained result for the multiplicities is expressed in terms of the so-called m-derangements numbers. Asymptotic convergence to randomness in the shuffling the m-cards is briefly analysed. We give a preliminary result on the spectrum of $L_{{}^{(m)}Ш_{1,n-1}}$ in the cyclotomic Hecke algebra $H(m,1,n)$, which is a deformation of the group algebra of the complex reflection group.

1. COMPLEX REFLECTION GROUPS $G(m,1,n)$

A finite complex reflection group is a finite subgroup of $GL_n(\mathbb{C})$ generated by complex reflections, that is, elements $\tau \in GL_n(\mathbb{C})$ of finite order such that $\mathrm{Ker}(\tau - id)$ is a hyperplane. The finite complex reflection groups have been classified by Shephard and Todd (1954) into an infinite family of groups

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